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Algebra 1

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Midterm

The grader cannot be expected to work his way through a sprawling mess of identities presented without a coherent narrative through line. If he can't make sense of it in finite time you could lose coherent narrative through line. If he can't make sense of it in finite time you could lose serious points. Coherent, readable exposition of your work is half the job in mathematics.

Problem 1 :

Consider a composite $\phi \circ \psi(x) = \phi(\psi(x))$ of two maps $X \xrightarrow{\psi} Y \xrightarrow{\phi} Z$. If the maps ϕ and ψ are injective/surjective/bijective, what can you say about $\phi \circ \psi$? Write I if the composite is injective but not surjective , S if it is subjective but not one-to-one, and B if is bijecitve. (These possibilities are mutually exclusive.)

1. **Enter one of these symbols in the following diagram if the composite $\phi \circ \psi$ always has that property.**
2. **Write $\times$ if $\phi \circ \psi$ does not always have one of these properties**

map ϕ

map ψ

	I	S	B
I			
S			
B			

You need not give any explanation for your answers, but will lose points for incorrect answer.

Problem 2 :
The rational numbers $\mathbb{Q}$ are constructed from the integers $\mathbb{Z}$ by imposing an RST relation $\sim$ on the set of pairs of elements of $\mathbb{Z}$,

$$X = \{(p,q) : p,q \in \mathbb{Z}, q \neq 0\} :$$

$$(p,q) \sim (r,s) \Leftrightarrow ps = qr$$

Then $\mathbb{Q}$ is the space of equivalence classes $[(p,q)]$ of fraction symbols. We impose algebraic operation $(+)$ and $(\cdot)$ in $\mathbb{Q}$ by defining

$$[(p,q)] + [(r,s)] = [(pq + qr, qs)] \text{ and } [(p,q)] \cdot [(r,s)] = [(pr,qs)]$$

From these definitions, prove that these operations well defined in spite of our use of representatives in their definitions, that means prove that $(p,q) \sim (p',q')$ and $(r,s) \sim (r',s')$ implies that $(ps + qr, qs) \sim (p's' + q'r', q's')$ and $(pr,qs) \sim (p'r', q's')$. (Note : We denote $p/q = [(p,q)]$.)

Problem 3 :
Let $(G,+)$ be an abelian group written in the additive notation. Integer $k \in \mathbb{Z}$ act on group element $x \in G$ by taking additive powers

$$\begin{cases} k \cdot x &= x + \cdots + x & (k \text{ times if } k > 0) \\ 0 \cdot x &= 0_G & (\text{ the additive identity element in } G) \\ -k \cdot x &= (-x) + \cdots + (-x) & (k > 0;\ -x = \text{additive inverse of } x) \end{cases}$$

If $a, b \in G$, prove that

1. **The subset $S = \mathbb{Z}a + \mathbb{Z}b = \{k \cdot a + l \cdot b : k, l \in \mathbb{Z}\}$ is a subgroup of $(G,+)$;**
2. **S is precisely the subgroup $H = <a, b>$ generated by a and b.**

Problem 4 :
Let H be a subgroup in a finite group G. If G is generated by a set S, so $G = <S>$, prove that H is a normal subgroup in G if and only if $sHs^{-1} \subseteq H$ for all $s \in S$.
Note : **Results of this sort are useful because it is easier to check an algebraic property for a small set of generators than to prove it holds for all elements of G.**

Problem 5 :
If G is a group that has no proper subgroups ($H \neq \{e\}$ and $H \neq \mathsf{G}$), In this exercise we will prove that :

1. *G must be cyclic and finite.*
2. *Either G is trivial or $\mathsf{G} \simeq (\mathbb{Z}/p\mathbb{Z}, +)$ for some prime $p > 1$.*

We do not assume G finite.

In order to prove this, please follow the questions above :

1. **If G is non-cyclic, give a proper subgroup of G ;**
2. **If G is cyclic and of infinite cardinality, to which well-known group G is it isomorphic (Theorem of structure of cyclic groups) ? Give then a proper subgroup of this very well-known group.**

We have now proven that G without proper subgroup implies that it is finite and cyclic.

3. **Suppose that G is finite, cyclic and non-trivial , to which well-known group G is it isomorphic (Theorem of structure of cyclic groups) ? If its cardinality is not prime, find a proper subgroup of this well-known group G.**

(Note that this complete the proof).